Sample Problems

Math 464

One notebook sized page of notes will be allowed on the final. You may write on both sides of the page. No electronic devices will be allowed. This set of problems only covers material since the midterm. The final will be comprehensive. There will be questions covering the first half and second half of the course. The exam is at **2:30 a.m. on Tuesday,, December 10**.

1. Suppose f is C^2 on an open interval in $I \subset \mathbb{R}$ and x_1, x_2, x_3 are distinct points of I. Prove that there exists $y \in I$ such that

   $$f(x_1)(x_3 - x_2) - f(x_2)(x_3 - x_1) + f(x_3)(x_2 - x_1) = \frac{1}{2} f''(y)(x_3 - x_2)(x_3 - x_1)(x_2 - x_1).$$

   Hint: Use the Lagrange interpolation formula and Rolle's theorem.

2. Find the weights of a quadrature formula on $[-1, 1]$

   $$Q(f) = Af(-1) + Bf(0) + Cf'(0) + Df(1),$$

   that has precision three (correctly integrates polynomials of degree three). Are numbers A, B, C, D unique?

3. The weights of a Gaussian quadrature formula are always positive. Prove that this is true for the Gaussian quadrature formula based on three points in the interval $[-1, 1]$. Hint: you don't need to explicitly find the Legendre polynomial of degree three.

4. Let $Q(f) = af(\frac{1}{4}) + bf(\frac{1}{2}) + cf(\frac{3}{4})$ be a quadrature formula on $[0, 1]$.

   a) If Q is to have precision at least 2, what are the values of a, b, c?

   b) What is the exact precision of the resulting quadrature?

5. Let $p(x)$ be the linear function that interpolates $\sin(x)$ at 0 and $\pi/2$. Prove that $|p(x) - \sin(x)| \leq \frac{1}{2}(\pi/4)^2$ on $[0, \pi/2]$.

6. Let A be a nonsingular 2×2 matrix. Prove that the ∞-norm condition number and 1-norm condition number are equal.

7. Find the polynomial p of degree 3 that satisfies the following conditions: $p(1) = 1, p'(1) = 2, p''(1) = 1, p(2) = 3$. Prove that the polynomial is unique.

8. a) What points x_0, x_1, x_2, x_3 should be used in $[0, 1]$ to minimize the error bound in polynomial interpolation at four points? Express your answer using the symbols Hint: Use $\cos x = \sqrt{\dfrac{1 + \cos 2x}{2}}$.

   b) What is the largest value attained by $(x - x_0)(x - x_1)(x - x_2)(x - x_3)$ in $[0, 1]$, where x_0, x_1, x_2, x_3 are the points chosen in part a)?

   In this problem, $[0, 1]$ is <u>not</u> the interval $[-1, 1]$.

9. Prove that
$$2T_n(x) = \left(x + i\sqrt{1 - x^2}\right)^n + \left(x - i\sqrt{1 - x^2}\right)^n,$$
for $n = 0, 1, 2$, where $i = \sqrt{-1}$ and T_n is the n^{th} Chebyshev polynomial. (This is formula is true for all n.)

10. The Chebyshev polynomial of degree 3 is $T_3(x) = 4x^3 - 3x$.

    (a) What is the interpolatory quadrature rule on the interval $[-1, 1]$ that uses the roots of T_3 as nodes?

    (b) What is the corresponding rule on any interval $[a, b]$? ("Translate" the interval.)

11. a) Let $p(x) = 2x^4 - 3x^3 + 5x^2 - 2x + 7$. *Use nested multiplication* to evaluate $p(2)$. Show the multiplications.

    b) Use nested multiplication to evaluate $p'(2)$. Show the multiplications. *Do <u>not</u> take the derivative of $p(x)$ and then use the polynomial $p'(x)$ to evaluate $p'(2)$.*

    c) Use nested multiplication again to evaluate $p''(2)$.

12. Let $p_3(x)$ be the unique polynomial of degree at most 3 that interpolates $f(x) = 2^{x+2}$ at the points $(x_0, x_1, x_2, x_3) = (-2, -1, 1, 2)$. Notice that 0 is <u>not</u> one of the points.

    a) Form the divided difference table for f at these points and write $p_3(x)$ in its Newton form.

    b) Give the best upper bound for $\|f - p_3\|_\infty$ on $[-2, 2]$.